

Multipliers of uniform topological algebras

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Abstract. Let E be a complete uniform topological algebra with Arens-Michael normed factors $(E_\alpha)_{\alpha \in \Lambda}$. Then $M(E) \cong \varprojlim M(E_\alpha)$ within an algebra isomorphism φ . If each factor E_α is complete, then every multiplier of E is continuous and φ is a topological algebra isomorphism where $M(E)$ is endowed with its seminorm topology.

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1. Preliminaries

A topological algebra is an algebra (over the complex field) which is also a Hausdorff topological vector space such that the multiplication is separately continuous. For a topological algebra E , we denote by $\Delta(E)$ the set of all nonzero continuous multiplicative linear functionals on E . An approximate identity in a topological algebra E is a net $(e_\omega)_{\omega \in \Omega}$ such that for each $x \in E$ we have $xe_\omega \rightarrow_\omega x$ and $e_\omega x \rightarrow_\omega x$. Let E be an algebra, a function $p : E \rightarrow [0, \infty[$ is called a pseudo-seminorm, if there exists $0 \leq k \leq 1$ such that $p(x+y) \leq p(x) + p(y)$, $p(\lambda x) = |\lambda|^k p(x)$ and $p(xy) \leq p(x)p(y)$ for all $x, y \in E$ and $\lambda \in \mathbb{C}$. k is called the homogeneity index of p . If $k = 1$, p is called a seminorm. A pseudo-seminorm p is a pseudo-norm, if $p(x) = 0$ implies $x = 0$. A locally m -pseudoconvex algebra is a topological algebra E whose topology is determined by a directed family $\{p_\alpha : \alpha \in \Lambda\}$ of pseudo-seminorms. For each $\alpha \in \Lambda$, $\ker(p_\alpha) = \{x \in E : p_\alpha(x) = 0\}$, the quotient algebra $E_\alpha = E/\ker(p_\alpha)$ is a pseudo-normed algebra in the pseudo-norm $\bar{p}_\alpha(x_\alpha) = p_\alpha(x)$, $x_\alpha = x + \ker(p_\alpha)$. Let $f_\alpha : E \rightarrow E_\alpha$, $f_\alpha(x) = x + \ker(p_\alpha) = x_\alpha$, be the quotient map, f_α is a continuous homomorphism from E onto E_α . We endow the set Λ with the partial order: $\alpha \leq \beta$, if and only, if $p_\alpha(x) \leq p_\beta(x)$ for all $x \in E$. Take $\alpha \leq \beta$ in Λ , since $\ker(p_\beta) \subset \ker(p_\alpha)$, we define the surjective continuous homomorphism $f_{\alpha\beta} : E_\beta \rightarrow E_\alpha$, $x_\beta = x + \ker(p_\beta) \rightarrow x_\alpha = x + \ker(p_\alpha)$. Thus $\{(E_\alpha, f_{\alpha\beta}), \alpha \leq \beta\}$ is a projective system of pseudo-normed algebras. We also define the algebra isomorphism (into) $\Phi : E \rightarrow \varprojlim E_\alpha$, $\Phi(x) = (f_\alpha(x))_{\alpha \in \Lambda}$, the canonical projections $\pi_\alpha : \prod_{\alpha \in \Lambda} E_\alpha \rightarrow E_\alpha$ and the restrictions to the projective limit $g_\alpha = \pi_\alpha / \varprojlim E_\alpha : \varprojlim E_\alpha \rightarrow E_\alpha$. Since $g_\alpha \circ \Phi = f_\alpha$ and the quotient map f_α is surjective, it follows that the map g_α is surjective, this proves that the projective system $\{(E_\alpha, f_{\alpha\beta}), \alpha \leq \beta\}$ is perfect in the sense of [5, Definition 2.10] (see also [2, Definition 2.7]). Thus, if E is a locally m -pseudoconvex algebra (not necessarily complete), then its generalized Arens-Michael projective

system $\{(E_\alpha, f_{\alpha\beta}), \alpha \leq \beta\}$ is perfect. If E is complete, then $E \cong \varprojlim E_\alpha$ within a topological algebra isomorphism.

A locally m -convex algebra is a topological algebra E whose topology is defined by a directed family $\{p_\alpha : \alpha \in \Lambda\}$ of seminorms. For each $\alpha \in \Lambda$, put $\Delta_\alpha(E) = \{f \in \Delta(E) : |f(x)| \leq p_\alpha(x), x \in E\}$. Let E be an algebra with involution $*$. A seminorm on E is called a C^* -seminorm if $p(x^*x) = p(x)^2$ for all $x \in E$. A complete locally m -convex $*$ -algebra $(E, (p_\alpha)_{\alpha \in \Lambda})$, for which each p_α is a C^* -seminorm, is called a locally C^* -algebra. A uniform seminorm on an algebra E is a seminorm p satisfying $p(x^2) = p(x)^2$ for all $x \in E$. A uniform topological algebra is a topological algebra whose topology is determined by a directed family of uniform seminorms. In that case, such a topological algebra is also named a *uniform locally convex algebra*. A uniform normed algebra is a normed algebra $(E, \|\cdot\|)$ such that $\|x^2\| = \|x\|^2$ for all $x \in E$.

An algebra E is called proper if for any $x \in E$, $xE = Ex = \{0\}$ implies $x = 0$. If E has identity, then E is proper. Moreover, a topological algebra with approximate identity is proper. Also, a (Hausdorff) uniform topological algebra is proper. Let E be an algebra, a map $T : E \rightarrow E$ is called a multiplier if $T(x)y = xT(y)$ for all $x, y \in E$. We denote by $M(E)$ the set of all multipliers of E . It is known that if E is a proper algebra, then any multiplier T of E is linear with the property $T(xy) = T(x)y = xT(y)$ for all $x, y \in E$, and $M(E)$ is a commutative algebra with the identity map I of E as its identity. Let $(E, \|\cdot\|)$ be a uniform normed algebra, and let $M_c(E)$ be the algebra of all continuous multipliers of E with the operator norm $\|\cdot\|_{op}$. It is known that $\|\cdot\|_{op}$ has the square property and the map $l : (E, \|\cdot\|) \rightarrow (M_c(E), \|\cdot\|_{op})$, $l(x)(y) = xy$, is an isometric isomorphism (into). For information on the multiplier algebra in non-normed topological algebras, see also [3] and [4].

In the sequel, we will need the following elementary result called the universal property of the quotient: Let X, Y, Z be vector spaces, $f : X \rightarrow Y$ and $g : X \rightarrow Z$ be linear maps. If the map g is surjective and $\ker(g) \subset \ker(f)$, then there exists a unique linear map $h : Z \rightarrow Y$ such that $f = h \circ g$.

2. Results

Proposition 2.1. *Let $(E, (p_\alpha)_{\alpha \in \Lambda})$ be a locally m -pseudoconvex algebra with proper pseudo-normed factors $(E_\alpha)_{\alpha \in \Lambda}$. The following assertions are equivalent:*

- (i) $T(\ker(f_\alpha)) \subset \ker(f_\alpha)$ for all $T \in M(E)$ and $\alpha \in \Lambda$;
- (ii) for each $T \in M(E)$, there exists a unique $(T_\alpha)_{\alpha \in \Lambda} \in \prod_{\alpha \in \Lambda} M(E_\alpha)$ such that $f_\alpha \circ T = T_\alpha \circ f_\alpha$ and $T_\alpha \circ f_{\alpha\beta} = f_{\alpha\beta} \circ T_\beta$ for all $\alpha \leq \beta$ in Λ . Furthermore, T is continuous if and only if T_α is continuous for all $\alpha \in \Lambda$.

Proof. Since the pseudo-normed factors $(E_\alpha)_{\alpha \in \Lambda}$ are proper, it follows that the algebra E is proper and so every multiplier of E (or E_α) is linear.

(ii) \Rightarrow (i) : If $T \in M(E)$ and $x \in \ker(f_\alpha)$, then $f_\alpha(T(x)) = T_\alpha(f_\alpha(x)) = 0$ and so $T(x) \in \ker(f_\alpha)$.

(i) \Rightarrow (ii) : Take $T \in M(E)$ and $\alpha \in \Lambda$. Since $T(\ker(f_\alpha)) \subset \ker(f_\alpha)$ and

by using the universal property of the quotient (see Preliminaries), there exists a unique linear map $T_\alpha : E_\alpha \rightarrow E_\alpha$ such that $f_\alpha \circ T = T_\alpha \circ f_\alpha$. Let $\alpha \in \Lambda$ and $x, y \in E$, $T_\alpha(f_\alpha(x)f_\alpha(y)) = T_\alpha(f_\alpha(xy)) = f_\alpha(T(xy)) = f_\alpha(xT(y)) = f_\alpha(x)f_\alpha(T(y)) = f_\alpha(x)T_\alpha(f_\alpha(y))$ and similarly on the other side, so T_α is a multiplier of E_α . Let $\alpha \leq \beta$ in Λ , we have $T_\alpha \circ f_\alpha = f_\alpha \circ T$, then $T_\alpha \circ f_{\alpha\beta} \circ f_\beta = f_{\alpha\beta} \circ f_\beta \circ T = f_{\alpha\beta} \circ T_\beta \circ f_\beta$, hence $T_\alpha \circ f_{\alpha\beta} = f_{\alpha\beta} \circ T_\beta$ since the quotient map f_β is surjective. Suppose that T is continuous. Let O_α be an open set in E_α , we have $f_\alpha^{-1}(T_\alpha^{-1}(O_\alpha)) = (T_\alpha \circ f_\alpha)^{-1}(O_\alpha) = (f_\alpha \circ T)^{-1}(O_\alpha)$ which is open in E since $f_\alpha \circ T$ is continuous, then $T_\alpha^{-1}(O_\alpha)$ is open in E_α . Conversely, suppose that T_α is continuous for all $\alpha \in \Lambda$. Since E is topologically isomorphic to a subalgebra of $\varinjlim E_\alpha$, T is continuous if and only if $f_\alpha \circ T$ is continuous for all $\alpha \in \Lambda$. Since $f_\alpha \circ T = T_\alpha \circ f_\alpha$ and T_α is continuous for all $\alpha \in \Lambda$, we deduce that T is continuous.

Proposition 2.2. *Let $(E, (p_\alpha)_{\alpha \in \Lambda})$ be a locally m -pseudoconvex algebra with proper pseudo-normed factors $(E_\alpha)_{\alpha \in \Lambda}$. The following assertions are equivalent:*
(j) $U(\ker(f_{\alpha\beta})) \subset \ker(f_{\alpha\beta})$ for all $U \in M(E_\beta)$ and $\alpha \leq \beta$ in Λ ;
(jj) there exists a unique projective system $\{(M(E_\alpha), h_{\alpha\beta}), \alpha \leq \beta\}$ such that $h_{\alpha\beta}(U) \circ f_{\alpha\beta} = f_{\alpha\beta} \circ U$ for all $U \in M(E_\beta)$ and $\alpha \leq \beta$ in Λ . Furthermore, if E_α is complete for all $\alpha \in \Lambda$, then $h_{\alpha\beta}$ is continuous for all $\alpha \leq \beta$ in Λ .

Proof. *(jj) \Rightarrow (j) :* Let $U \in M(E_\beta)$ and $x_\beta \in \ker(f_{\alpha\beta})$, then $f_{\alpha\beta}(U(x_\beta)) = h_{\alpha\beta}(U)(f_{\alpha\beta}(x_\beta)) = 0$ and so $U(x_\beta) \in \ker(f_{\alpha\beta})$.
(j) \Rightarrow (jj) : Let $\alpha \leq \beta$ in Λ and $U \in M(E_\beta)$. Since $U(\ker(f_{\alpha\beta})) \subset \ker(f_{\alpha\beta})$ and by using the universal property of the quotient (see Preliminaries), there exists a unique linear map $V : E_\alpha \rightarrow E_\alpha$ such that $V \circ f_{\alpha\beta} = f_{\alpha\beta} \circ U$. Let $x_\alpha = x + \ker(p_\alpha)$, $y_\alpha = y + \ker(p_\alpha) \in E_\alpha$ where $x, y \in E$. Put $x_\beta = x + \ker(p_\beta)$ and $y_\beta = y + \ker(p_\beta)$, clearly $x_\beta, y_\beta \in E_\beta$. By definition of the map $f_{\alpha\beta}$, we get $f_{\alpha\beta}(x_\beta) = x_\alpha$ and $f_{\alpha\beta}(y_\beta) = y_\alpha$. We have $V(x_\alpha y_\alpha) = V(f_{\alpha\beta}(x_\beta)f_{\alpha\beta}(y_\beta)) = V(f_{\alpha\beta}(x_\beta y_\beta)) = f_{\alpha\beta}(U(x_\beta y_\beta)) = f_{\alpha\beta}(x_\beta U(y_\beta)) = f_{\alpha\beta}(x_\beta)f_{\alpha\beta}(U(y_\beta)) = f_{\alpha\beta}(x_\beta)V(f_{\alpha\beta}(y_\beta)) = x_\alpha V(y_\alpha)$ and similarly on the other side, so V is a multiplier of E_α . This shows the existence of the map $h_{\alpha\beta} : M(E_\beta) \rightarrow M(E_\alpha)$ such that $h_{\alpha\beta}(U) \circ f_{\alpha\beta} = f_{\alpha\beta} \circ U$ for all $U \in M(E_\beta)$ and $\alpha \leq \beta$ in Λ . Let $\alpha \leq \beta$ in Λ , $U_1, U_2 \in M(E_\beta)$ and $\lambda \in \mathbb{C}$, $h_{\alpha\beta}(U_1 + \lambda U_2) \circ f_{\alpha\beta} = f_{\alpha\beta} \circ (U_1 + \lambda U_2) = (f_{\alpha\beta} \circ U_1) + \lambda(f_{\alpha\beta} \circ U_2) = h_{\alpha\beta}(U_1) \circ f_{\alpha\beta} + \lambda h_{\alpha\beta}(U_2) \circ f_{\alpha\beta} = (h_{\alpha\beta}(U_1) + \lambda h_{\alpha\beta}(U_2)) \circ f_{\alpha\beta}$, hence $h_{\alpha\beta}(U_1 + \lambda U_2) = h_{\alpha\beta}(U_1) + \lambda h_{\alpha\beta}(U_2)$ since $f_{\alpha\beta}$ is surjective. Also, $h_{\alpha\beta}(U_1 \circ U_2) \circ f_{\alpha\beta} = f_{\alpha\beta} \circ U_1 \circ U_2 = h_{\alpha\beta}(U_1) \circ f_{\alpha\beta} \circ U_2 = h_{\alpha\beta}(U_1) \circ h_{\alpha\beta}(U_2) \circ f_{\alpha\beta}$, then $h_{\alpha\beta}(U_1 \circ U_2) = h_{\alpha\beta}(U_1) \circ h_{\alpha\beta}(U_2)$ since $f_{\alpha\beta}$ is surjective. Let $\alpha \leq \beta \leq \gamma$ in Λ and $W \in M(E_\gamma)$, $(h_{\alpha\beta} \circ h_{\beta\gamma})(W) \circ f_{\alpha\gamma} = h_{\alpha\beta}(h_{\beta\gamma}(W)) \circ f_{\alpha\beta} \circ f_{\beta\gamma} = f_{\alpha\beta} \circ h_{\beta\gamma}(W) \circ f_{\beta\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma} \circ W = f_{\alpha\gamma} \circ W = h_{\alpha\gamma}(W) \circ f_{\alpha\gamma}$, consequently $(h_{\alpha\beta} \circ h_{\beta\gamma})(W) = h_{\alpha\gamma}(W)$ since $f_{\alpha\gamma}$ is surjective. Thus $h_{\alpha\beta} \circ h_{\beta\gamma} = h_{\alpha\gamma}$. Let $\alpha \in \Lambda$, if E_α is complete, then every multiplier of E_α is continuous. Now by assuming that E_α is complete for all $\alpha \in \Lambda$, we will show that $h_{\alpha\beta}$ is continuous for all $\alpha \leq \beta$ in Λ (see also, the proof of Theorem 2.12 in [5]). For $\alpha \in \Lambda$ and $r \gtrsim 0$, let $B_\alpha(0, r) = \{x_\alpha \in E_\alpha : \overline{p}_\alpha(x_\alpha) \leq r\}$. We denote by $\|\cdot\|_\alpha$ the

operator pseudo-norm on $M(E_\alpha)$. Let $\alpha \leq \beta$ in Λ , $f_{\alpha\beta}$ is open by the open mapping theorem, so there is $\lambda \geq 0$ such that $\lambda B_\alpha(0, 1) \subset f_{\alpha\beta}(B_\beta(0, 1))$ i.e. $B_\alpha(0, 1) \subset f_{\alpha\beta}(B_\beta(0, r))$ where $r = \lambda^{-k_\beta}$ and k_β is the homogeneity index of \bar{p}_β . Let $U \in M(E_\beta)$,

$$\begin{aligned} \|h_{\alpha\beta}(U)\|_\alpha &= \sup \{\bar{p}_\alpha(h_{\alpha\beta}(U)(f_\alpha(x))) : f_\alpha(x) \in B_\alpha(0, 1)\} \\ &\leq \sup \{\bar{p}_\alpha(h_{\alpha\beta}(U)(f_{\alpha\beta}(f_\beta(x)))) : f_\beta(x) \in B_\beta(0, r)\} \\ &= \sup \{\bar{p}_\alpha(f_{\alpha\beta}(U(f_\beta(x)))) : f_\beta(x) \in B_\beta(0, r)\} \\ &\leq \sup \{\bar{p}_\beta(U(f_\beta(x))) : f_\beta(x) \in B_\beta(0, r)\} \\ &\leq \sup \{\|U\|_\beta \bar{p}_\beta(f_\beta(x)) : f_\beta(x) \in B_\beta(0, r)\} \\ &= r\|U\|_\beta. \end{aligned}$$

Therefore $h_{\alpha\beta}$ is continuous.

Theorem 2.3. *Let $(E, (p_\alpha)_{\alpha \in \Lambda})$ be a complete locally m -pseudoconvex algebra with proper pseudo-normed factors $(E_\alpha)_{\alpha \in \Lambda}$. Assume that E satisfies conditions (i) and (j). Then $M(E) \cong \varprojlim M(E_\alpha)$ within an algebra isomorphism φ . Furthermore, if each factor E_α is complete, then every multiplier of E is continuous and φ is a topological algebra isomorphism where $M(E)$ is endowed with its pseudo-seminorm topology.*

Proof. Take $T \in M(E)$. By Propositions 2.1 and 2.2, $(T_\alpha)_{\alpha \in \Lambda} \in \prod_{\alpha \in \Lambda} M(E_\alpha)$, $T_\alpha \circ f_{\alpha\beta} = f_{\alpha\beta} \circ T_\beta$ and $h_{\alpha\beta}(T_\beta) \circ f_{\alpha\beta} = f_{\alpha\beta} \circ T_\beta$ for all $\alpha \leq \beta$ in Λ . Hence $h_{\alpha\beta}(T_\beta) \circ f_{\alpha\beta} = T_\alpha \circ f_{\alpha\beta}$ and consequently $h_{\alpha\beta}(T_\beta) = T_\alpha$ since the map $f_{\alpha\beta}$ is surjective. This shows that $(T_\alpha)_{\alpha \in \Lambda} \in \varprojlim M(E_\alpha)$. Thus the map $\varphi : M(E) \rightarrow \varprojlim M(E_\alpha)$, $T \rightarrow (T_\alpha)_{\alpha \in \Lambda}$, is well defined. We will show that φ is an algebra isomorphism. Let $T, S \in M(E)$ and $\lambda \in \mathbb{C}$, $T_\alpha \circ f_\alpha = f_\alpha \circ T$ and $S_\alpha \circ f_\alpha = f_\alpha \circ S$, then $(T_\alpha + \lambda S_\alpha) \circ f_\alpha = f_\alpha \circ (T + \lambda S)$, so $(T + \lambda S)_\alpha = T_\alpha + \lambda S_\alpha$ by Proposition 2.1. Also, $T_\alpha \circ S_\alpha \circ f_\alpha = T_\alpha \circ f_\alpha \circ S = f_\alpha \circ T \circ S$, hence $(T \circ S)_\alpha = T_\alpha \circ S_\alpha$ by Proposition 2.1. Let $T \in M(E)$, if $T_\alpha = 0$ for all $\alpha \in \Lambda$, then $f_\alpha \circ T = T_\alpha \circ f_\alpha = 0$ for all $\alpha \in \Lambda$ and consequently $T = 0$. Let $(U_\alpha)_{\alpha \in \Lambda} \in \varprojlim M(E_\alpha)$ and define the map $T = \Phi^{-1} \circ \varprojlim U_\alpha \circ \Phi : E \rightarrow E$ where $\varprojlim U_\alpha$ is the multiplier of $\varprojlim E_\alpha$ defined by $(\varprojlim U_\alpha)(x)_\alpha = (U_\alpha(x))_\alpha$ and $\Phi : E \rightarrow \varprojlim E_\alpha$ is the topological algebra isomorphism given by $\Phi(x) = (f_\alpha(x))_\alpha$. Clearly T is a multiplier of E , also $f_\alpha \circ T = f_\alpha \circ \Phi^{-1} \circ \varprojlim U_\alpha \circ \Phi = U_\alpha \circ f_\alpha$ for all $\alpha \in \Lambda$, so $\varphi(T) = (U_\alpha)_\alpha$. If E_α is complete for all $\alpha \in \Lambda$, then every multiplier of E_α is continuous, hence every multiplier of E is continuous by Proposition 2.1. The pseudo-seminorm topology on $M(E)$ is the topology defined by the family of pseudo-seminorms $q_\alpha(T) = \|T_\alpha\|_\alpha$, $\alpha \in \Lambda$, so φ is a topological algebra isomorphism.

Proposition 2.4. *Let $(E, (p_\alpha)_{\alpha \in \Lambda})$ be a locally m -pseudoconvex algebra with approximate identity $(e_\omega)_{\omega \in \Omega}$. Then E satisfies conditions (i) and (j).*

Proof. Let $T \in M(E)$, $x \in \ker(f_\alpha)$ and $\omega \in \Omega$,

$$\begin{aligned} f_\alpha(T(x)) &= f_\alpha(T(x - xe_\omega + xe_\omega)) \\ &= f_\alpha(T(x) - T(xe_\omega)) + f_\alpha(T(xe_\omega)) = f_\alpha(T(x) - T(x)e_\omega) + f_\alpha(xT(e_\omega)) \\ &= f_\alpha(T(x) - T(x)e_\omega) + f_\alpha(x)f_\alpha(T(e_\omega)) = f_\alpha(T(x) - T(x)e_\omega). \end{aligned}$$

Since $T(x)e_\omega \rightarrow_\omega T(x)$ and f_α is continuous, we deduce that $f_\alpha(T(x)) = 0$. Now we will show that $U(\ker(f_{\alpha\beta})) \subset \ker(f_{\alpha\beta})$ for all $U \in M(E_\beta)$ and $\alpha \leq \beta$ in Λ . Since $(e_\omega)_{\omega \in \Omega}$ is an approximate identity in E and $f_\beta : E \rightarrow E_\beta$ is a surjective continuous homomorphism, it follows that $(f_\beta(e_\omega))_{\omega \in \Omega}$ is an approximate identity in E_β (see [8, Theorem 4.1]). Let $U \in M(E_\beta)$, $x_\beta \in \ker(f_{\alpha\beta})$ and $\omega \in \Omega$,

$$\begin{aligned} f_{\alpha\beta}(U(x_\beta)) &= f_{\alpha\beta}(U(x_\beta - x_\beta f_\beta(e_\omega) + x_\beta f_\beta(e_\omega))) \\ &= f_{\alpha\beta}(U(x_\beta) - U(x_\beta f_\beta(e_\omega))) + f_{\alpha\beta}(U(x_\beta f_\beta(e_\omega))) \\ &= f_{\alpha\beta}(U(x_\beta) - U(x_\beta) f_\beta(e_\omega)) + f_{\alpha\beta}(x_\beta U(f_\beta(e_\omega))) \\ &= f_{\alpha\beta}(U(x_\beta) - U(x_\beta) f_\beta(e_\omega)) + f_{\alpha\beta}(x_\beta) f_{\alpha\beta}(U(f_\beta(e_\omega))) \\ &= f_{\alpha\beta}(U(x_\beta) - U(x_\beta) f_\beta(e_\omega)). \end{aligned}$$

Since $U(x_\beta) f_\beta(e_\omega) \rightarrow_\omega U(x_\beta)$ and $f_{\alpha\beta}$ is continuous, we deduce that $f_{\alpha\beta}(U(x_\beta)) = 0$.

Corollary 2.5. [5, Theorems 2.6 and 2.12] *Let $(E, (p_\alpha)_{\alpha \in \Lambda})$ be a complete locally m -pseudoconvex algebra with approximate identity. Suppose that each factor $E_\alpha = E/\ker(p_\alpha)$ in the generalized Arens-Michael decomposition of E is complete. Then every multiplier of E is continuous and $M(E) \cong \varprojlim M(E_\alpha)$ within a topological algebra isomorphism where $M(E)$ is endowed with its pseudo-seminorm topology.*

Proof. It follows from Theorem 2.3 and Proposition 2.4.

Corollary 2.6. *Let $(E, (p_\alpha)_{\alpha \in \Lambda})$ be a locally C^* -algebra. Then every multiplier of E is continuous and $M(E) \cong \varprojlim M(E_\alpha)$ within a topological algebra isomorphism where $M(E)$ is endowed with its seminorm topology.*

Proof. By [7, Theorem 2.6] and [10, Corollary 1.12], E has an approximate identity and each factor E_α is complete.

Now we will describe multiplier algebras of complete uniform topological algebras.

Proposition 2.7. *Let $(E, (p_\alpha)_{\alpha \in \Lambda})$ be a uniform topological algebra. Then $\ker(f_\alpha) = \cap \{\ker(\chi) : \chi \in \Delta_\alpha(E)\}$ for all $\alpha \in \Lambda$ and $\ker(f_{\alpha\beta}) = \cap \{\ker(\mu \circ f_{\alpha\beta}) : \mu \in \Delta(E_\alpha)\}$ for all $\alpha \leq \beta$ in Λ .*

Proof. Show first that $\Delta_\alpha(E)$ and $\Delta(E_\alpha)$ are non empty sets. Let F_α be the completion of $(E_\alpha, \overline{p}_\alpha)$, F_α is a uniform Banach algebra. By [8, Lemma 5.1], F_α is commutative and semisimple. Then $\Delta(F_\alpha)$ is a non empty set since F_α is not a radical algebra, hence $\Delta_\alpha(E)$ and $\Delta(E_\alpha)$ are non empty sets (see [9, Proposition 7.5]).

By [1, Theorem 6], $p_\alpha(x) = \sup \{|\chi(x)| : \chi \in \Delta_\alpha(E)\}$ for all $x \in E$ and $\alpha \in \Lambda$, then $\ker(f_\alpha) = \ker(p_\alpha) = \cap \{\ker(\chi) : \chi \in \Delta_\alpha(E)\}$ for all $\alpha \in \Lambda$. Let $\alpha \leq \beta$ in Λ and $x_\beta \in E_\beta$,
 $x_\beta \in \ker(f_{\alpha\beta}) \Leftrightarrow f_{\alpha\beta}(x_\beta) = 0$

$$\begin{aligned} &\Leftrightarrow \mu(f_{\alpha\beta}(x_\beta)) = 0 \text{ for all } \mu \in \Delta(E_\alpha) \\ &\Leftrightarrow x_\beta \in \cap \{ker(\mu \circ f_{\alpha\beta}) : \mu \in \Delta(E_\alpha)\}. \end{aligned}$$

Proposition 2.8. *Let $(E, (p_\alpha)_{\alpha \in \Lambda})$ be a uniform topological algebra. Then E satisfies conditions (i) and (j).*

Proof. By Proposition 2.7, $ker(f_\alpha) = \cap \{ker(\chi) : \chi \in \Delta_\alpha(E)\}$ for all $\alpha \in \Lambda$. If T is a multiplier of E , then $T(ker(\chi)) \subset ker(\chi)$ for all $\chi \in \Delta_\alpha(E)$ by [6, Theorem 2.9] and [8, Lemma 5.1], so
 $T(ker(f_\alpha)) = T(\cap \{ker(\chi) : \chi \in \Delta_\alpha(E)\})$
 $\subset \cap \{T(ker(\chi)) : \chi \in \Delta_\alpha(E)\}$
 $\subset \cap \{ker(\chi) : \chi \in \Delta_\alpha(E)\} = ker(f_\alpha)$.
By Proposition 2.7, $ker(f_{\alpha\beta}) = \cap \{ker(\mu \circ f_{\alpha\beta}) : \mu \in \Delta(E_\alpha)\}$ for all $\alpha \leq \beta$ in Λ . If U is a multiplier of E_β , then $U(ker(\delta)) \subset ker(\delta)$ for all $\delta \in \Delta(E_\beta)$ by [6, Theorem 2.9] and [8, Lemma 5.1], so $U(ker(\mu \circ f_{\alpha\beta})) \subset ker(\mu \circ f_{\alpha\beta})$ for all $\mu \in \Delta(E_\alpha)$, and consequently
 $U(ker(f_{\alpha\beta})) = U(\cap \{ker(\mu \circ f_{\alpha\beta}) : \mu \in \Delta(E_\alpha)\})$
 $\subset \cap \{U(ker(\mu \circ f_{\alpha\beta})) : \mu \in \Delta(E_\alpha)\}$
 $\subset \cap \{ker(\mu \circ f_{\alpha\beta}) : \mu \in \Delta(E_\alpha)\} = ker(f_{\alpha\beta})$.

Theorem 2.9. *Let $(E, (p_\alpha)_{\alpha \in \Lambda})$ be a complete uniform topological algebra. Then $M(E) \cong \varprojlim M(E_\alpha)$ within an algebra isomorphism φ . Furthermore, if each factor E_α is complete, then every multiplier of E is continuous and φ is a topological algebra isomorphism where $M(E)$ is endowed with its seminorm topology.*

Proof. It follows from Theorem 2.3 and Proposition 2.8.

Remark. Let $(E, (p_\alpha)_{\alpha \in \Lambda})$ be a complete uniform topological algebra which is also a symmetric $*$ -algebra. Then $\chi(x^*) = \overline{\chi(x)}$ for all $x \in E$ and $\chi \in \Delta(E)$ (see [9, Lemma 6.4]). Take $x \in E$ and $\alpha \in \Lambda$. By [1, Theorem 6],
 $p_\alpha(x^*x) = \sup \{|\chi(x^*x)| : \chi \in \Delta_\alpha(E)\} = \sup \{|\chi(x)|^2 : \chi \in \Delta_\alpha(E)\}$
 $= (\sup \{|\chi(x)| : \chi \in \Delta_\alpha(E)\})^2 = p_\alpha(x)^2$.
Therefore $(E, (p_\alpha)_{\alpha \in \Lambda})$ is a locally C^* -algebra, and so each factor E_α is complete.

As an application of previous results, we deduce the Arhippainen unitization theorem [1, Theorem 4] on uniform topological algebras.

Proposition 2.10. *Let $(E, (p_\alpha)_{\alpha \in \Lambda})$ be a uniform topological algebra, and let $M_c(E)$ be the algebra of all continuous multipliers of E . Then there is a family of seminorms $(q_\alpha)_{\alpha \in \Lambda}$ on $M_c(E)$ such that*

1. $(M_c(E), (q_\alpha)_{\alpha \in \Lambda})$ is a uniform topological algebra;
2. the map $L : E \rightarrow M_c(E)$, $L(x)(y) = xy$, is an algebra isomorphism (into) and $q_\alpha(L(x)) = p_\alpha(x)$ for all $x \in E$ and $\alpha \in \Lambda$.

Proof. 1. By Propositions 2.1, 2.2 and 2.8, we define the map $\psi : M_c(E) \rightarrow \varprojlim M_c(E_\alpha)$, $T \rightarrow (T_\alpha)_{\alpha \in \Lambda}$. As in the proof of Theorem 2.3, ψ is an injective homomorphism. We endow $M_c(E)$ with the topology defined by the family of seminorms $q_\alpha(T) = \|T_\alpha\|_\alpha$, $\alpha \in \Lambda$, where $\|\cdot\|_\alpha$ is the operator norm on $M_c(E_\alpha)$. Let $T \in M_c(E)$, $q_\alpha(T^2) = \|(T^2)_\alpha\|_\alpha = \|(T_\alpha)^2\|_\alpha = \|T_\alpha\|_\alpha^2 = q_\alpha(T)^2$ since $\|\cdot\|_\alpha$ has the square property. Let $T \in M_c(E)$ with $q_\alpha(T) = 0$ for all $\alpha \in \Lambda$, then $T_\alpha = 0$ for all $\alpha \in \Lambda$, so $T = 0$ since ψ is injective.

2. Since E is proper, L is an algebra isomorphism (into). Let $x \in E$ and $\alpha \in \Lambda$, $(L(x))_\alpha \circ f_\alpha = f_\alpha \circ L(x)$, then $(L(x))_\alpha(f_\alpha(y)) = (f_\alpha \circ L(x))(y) = f_\alpha(xy) = f_\alpha(x)f_\alpha(y)$ for all $y \in E$. Since the map $l : (E_\alpha, \overline{p}_\alpha) \rightarrow (M_c(E_\alpha), \|\cdot\|_\alpha)$, $l(x_\alpha)(y_\alpha) = x_\alpha y_\alpha$, is an isometric isomorphism (into), it follows that $\|(L(x))_\alpha\|_\alpha = \overline{p}_\alpha(f_\alpha(x)) = p_\alpha(x)$, so $q_\alpha(L(x)) = p_\alpha(x)$.

Proposition 2.11. *Let E be a uniform topological algebra without unit, and let E_e be the algebra obtained from E by adjoining the unit. Then the map $g : E_e \rightarrow M_c(E)$, $g((x, \lambda)) = L(x) + \lambda I$ is an algebra isomorphism (into).*

Proof. It is easy to show that g is an algebra homomorphism. Let $(x, \lambda) \in E_e$ with $g((x, \lambda)) = 0$, then $L(x) = -\lambda I$. Suppose $\lambda \neq 0$, $I = -\lambda^{-1}L(x) = L(-\lambda^{-1}x)$, so $-\lambda^{-1}x$ is a left unit in E . Since E is commutative, $-\lambda^{-1}x$ is a unit in E , a contradiction. Thus $L(x) = 0$ and consequently $x = 0$ since E is proper.

Corollary 2.12. [1, Theorem 4] *Let $(E, (p_\alpha)_{\alpha \in \Lambda})$ be a uniform topological algebra without unit. Then there is a family of seminorms $(s_\alpha)_{\alpha \in \Lambda}$ on E_e such that $(E_e, (s_\alpha)_{\alpha \in \Lambda})$ is a uniform topological algebra and $s_\alpha((x, 0)) = p_\alpha(x)$ for all $x \in E$ and $\alpha \in \Lambda$.*

Proof. For each $\alpha \in \Lambda$, we define a seminorm on E_e by $s_\alpha((x, \lambda)) = q_\alpha(L(x) + \lambda I)$ for all $x \in E$ and $\lambda \in \mathbb{C}$. By Propositions 2.10 and 2.11, $(E_e, (s_\alpha)_{\alpha \in \Lambda})$ is a uniform topological algebra and $s_\alpha((x, 0)) = q_\alpha(L(x)) = p_\alpha(x)$ for all $x \in E$.

Remark. We have $s_\alpha((x, \lambda)) = q_\alpha(L(x) + \lambda I) \leq q_\alpha(L(x)) + |\lambda|q_\alpha(I) = p_\alpha(x) + |\lambda|$ for all $x \in E$ and $\lambda \in \mathbb{C}$. This shows that the topology on E_e defined by the family of seminorms $(s_\alpha)_{\alpha \in \Lambda}$ is weaker than the usual topology on E_e defined by the family of seminorms $(\tilde{p}_\alpha)_{\alpha \in \Lambda}$ where $\tilde{p}_\alpha((x, \lambda)) = p_\alpha(x) + |\lambda|$.

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